

Adding fractions

What you should know

How to add fractions.
What a rational number is.

New idea

Unit fractions are those fractions with 1 as the numerator and a positive integer as the denominator. These are the first few unit fractions: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \dots$

Task: Using only unit fractions

First, use a spreadsheet to investigate these questions. The spreadsheet itself won't prove anything but it should help you to form an opinion of the answers to the first three questions.

- Starting with $\frac{1}{1}$ and adding on each unit fraction in turn, is the sum ever an integer?
- Starting with $\frac{1}{1}$ and adding on each unit fraction in turn, does the sum grow without limit or is there a ceiling beyond which it cannot pass?
- What happens if you add all those unit fractions with denominators that are positive integer powers of a fixed number, such as $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \dots$?
- Any rational number can be written as the sum of **different** unit fractions. How can you prove this?

Next, think about the four boxes below. These are hints to the questions above. You need to turn these into full explanations.

$\frac{1}{1} + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$ $> \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots$ $= \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$	$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{147}{60}$ <p>Try other series. Do you always get an odd numerator and an even denominator? Why?</p>
<p>Write $\frac{1}{n} - \frac{1}{n+1}$ as a single fraction.</p> <p>You can use the result to split a fraction into unit fractions.</p>	$S = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$ <p>Then $\frac{1}{3}S = \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \dots$</p> <p>Therefore $S = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots = \frac{1}{3} + \frac{1}{3}S$</p> <p>So $\frac{2}{3}S = \frac{1}{3}$ which means $S = \frac{1}{2}$.</p>

Take it further

Find out about the Egyptians' use of unit fractions, the harmonic series and geometric progressions.

Where this goes next

At A level you will study properties of series in Core Mathematics.