Adding fractions

What you should know

How to add fractions.
What a rational number is.

New idea

Unit fractions are those fractions with 1 as the numerator and a positive integer as the denominator. These are the first few unit fractions: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \ldots$.

Task: Using only unit fractions

First, use a spreadsheet to investigate these questions. The spreadsheet itself won't prove anything but it should help you to form an opinion of the answers to the first three questions.

- Starting with $\frac{1}{1}$ and adding on each unit fraction in turn, is the sum ever an integer?
- Starting with $\frac{1}{1}$ and adding on each unit fraction in turn, does the sum grow without limit or is there a ceiling beyond which it cannot pass?
- What happens if you add all those unit fractions with denominators that are positive integer powers of a fixed number, such as $\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\ldots$?
- Any rational number can be written as the sum of **different** unit fractions. How can you prove this?

Next, think about the four boxes below. These are hints to the questions above. You need to turn these into full explanations.

$\frac{1}{1}+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ldots$ $>\frac{1}{1}+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\ldots$ $=\frac{1}{1}+\frac{1}{2}+\quad\frac{1}{2}\quad+\qquad\frac{1}{2}\qquad+\ldots$	$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{147}{60}$ Try other series. Do you always get an odd numerator and an even denominator? Why?
Write $\frac{1}{n}-\frac{1}{n+1}$ as a single fraction. You can use the result to split a fraction into unit fractions.	$S=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots$ Then $\frac{1}{3}S=\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\ldots$ Therefore $S=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots=\frac{1}{3}+\frac{1}{3}S$ So $\frac{2}{3}S=\frac{1}{3}$ which means $S=\frac{1}{2}$.

Take it further

Find out about the Egyptians' use of unit fractions, the harmonic series and geometric progressions.

Where this goes next

At A level you will study properties of series in Core Mathematics.

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