FM Network December 2008 Newsletter
Recreational Mathematics problem Solutions

In the December 2008 FM Network Newsletter we posed three questions that were from the MEI 'Maths item of the month'. Discussion of the three questions and possible solutions can be found in the following pages.

1. Some positive numbers add up to 19. What is their maximum product?

When asked this question, students usually come up with $9 \times 10 = 90$ and, soon after, $9.5 \times 9.5 = 90.25$. A while later some will realise that you can have more than two numbers and the product quickly increases from $6 \times 6 \times 7 = 252$ to $3 \times 3 \times 3 \times 3 \times 3 \times 2 \times 2 = 972$. It's interesting to think why this is the largest answer using integers.

Can this be improved if we don't restrict ourselves to integers?

Since $6 \times \frac{19}{6} = 19$ then $\left(\frac{19}{6}\right)^6 \approx 1008$ is a possible product as are $\left(\frac{19}{7}\right)^7 \approx 1085.398$ and $\left(\frac{19}{8}\right)^8 \approx 1012$.

So is $\left(\frac{19}{7}\right)^7$ the biggest? It would appear that if all the numbers are the same, and you might be able to think of a symmetry argument which shows that the maximum product would necessarily consist of lots of the same number, then this is the maximum. Or is it?

2. To find the tangent to a parabola at a point P:

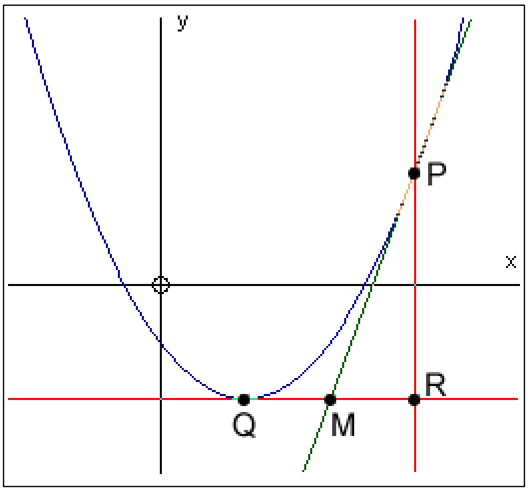
Draw a vertical line through P.

Draw a horizontal line through Q, the vertex of the parabola.

R is the intersection of these two lines.

M is the mid-point of Q and R.

MP is the tangent to the parabola at P.



Can you *prove* that MP is the tangent to the parabola?

Define the coordinates of Q as (a, b).

The equation of the quadratic is then $y = k(x-a)^2 + b$.

Define the *x*-coordinate of P as *p*.
The coordinates of P are $(p, k(p-a)^2+b)$.

The horizontal line has equation *y* = *b*.
The vertical line has equation *x* = *p*.
R has coordinates: (p,b).
M has coordinates: $(\frac{p+a}{2}, b)$.

By geometry the line through M and P has gradient:

$$\frac{k(p-a)^2+b-b}{\left(p-\frac{p+a}{2}\right)} = \frac{k(p-a)^2}{\left(\frac{2p-p-a}{2}\right)}$$
$$= \frac{2k(p-a)^2}{p-a}$$
$$= 2k(p-a)$$

and passes through P (by definition).

By calculus the derivative of $y = k(x-a)^2+b$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = 2k(x-a)$.
So the gradient of the tangent to the curve at P is $2k(p-a)$
and passes through P by definition.

Therefore the line through MP is the same as the tangent to the curve at the point P.

3. What is the smallest positive integer that cannot be expressed using exactly three twos and any mathematical operations you wish?

Here are some examples: $4 = \sqrt{2^{2\times 2}}$, $5 = \frac{2}{.2\times 2}$, $6 = {}^{2+2}C_2$, $7 = \frac{2}{.\dot{2}} - 2$

Surprisingly, any positive integer can be expressed using nested square roots as follows:

$$-\log_2\left(\log_2 \sqrt{2}\right) = -\log_2\left(\log_2 2^{\frac{1}{2}}\right) = -\log_2\left(\frac{1}{2}\right) = -(-1) = 1$$

$$-\log_2\left(\log_2 \sqrt{\sqrt{2}}\right) = -\log_2\left(\log_2 2^{\frac{1}{2^2}}\right) = -\log_2\left(\frac{1}{2^2}\right) = -(-2) = 2$$

$$-\log_2\left(\log_2 \sqrt{\sqrt{\sqrt{2}}}\right) = -\log_2\left(\log_2 2^{\frac{1}{2^3}}\right) = -\log_2\left(\frac{1}{2^3}\right) = -(-3) = 3$$

In general, $N = -\log_2\left(\log_2 x\right)$ where x is the number obtained by taking N successive square roots of 2.

EXTRA

Question 1 was also set to teachers on the TAM course in Chichester on a day when they were focusing on differentiation and, in particular, the chain and product rules.

One of these teachers, Simon, later gave detail of how he had approached the problem and it was as follows.

Assume all the numbers are the same. Let this number be $\frac{19}{x}$ and so we have x of them.

The product, $P = \left(\frac{19}{x}\right)^x$. Taking natural logarithms of both sides we have

$$\ln P = \ln\left(\frac{19}{x}\right)^x = x\ln\left(\frac{19}{x}\right) = x\left(\ln 19 - \ln x\right)$$

If we want to find the maximum value of P as x changes we need to solve $\frac{\mathrm{d}P}{\mathrm{d}x} = 0$. The problem is that we can differentiate the right hand side of $\ln P = x\left(\ln 19 - \ln x\right)$ but the left hand side looks awkward.

To see how to do it, let $z = \ln P$.

Now $\frac{\mathrm{d}z}{\mathrm{d}P} = \frac{1}{P}$ and by the chain rule $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}P} \times \frac{\mathrm{d}P}{\mathrm{d}x} = \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}x}$.

So differentiating $z = \ln P = x\left(\ln 19 - \ln x\right)$ with respect to x we get:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(\ln 19 - \ln x\right)\right)$$

$$\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}x} = \left(\ln 19 - \ln x\right) + x\left(-\frac{1}{x}\right)$$

Making $\frac{\mathrm{d}P}{\mathrm{d}x}$ the subject and simplifying:

$$\frac{\mathrm{d}P}{\mathrm{d}x} = P\left(\ln 19 - \ln x - 1\right)$$

The maximum value occurs when $\frac{\mathrm{d}P}{\mathrm{d}x} = 0$. Since we know the product $P \neq 0$ then we need to solve $\ln 19 - \ln x - 1 = 0$. This is equivalent to $\ln \frac{19}{x} = 1$ and so we require $\frac{19}{x} = \mathrm{e}$.

Therefore the maximum product is $P = \left(\frac{19}{x}\right)^x = \mathrm{e}^{\frac{19}{\mathrm{e}}} \approx 1085.406$, slightly bigger than $\left(\frac{19}{7}\right)^7 \approx 1085.398$.

But there's a problem. We were asked to split 19 up into 'some numbers'. Doesn't this mean that we have to have, for example, 5 or 6 or 7 numbers rather than $\frac{19}{\mathrm{e}} \approx 6.9897$ lots of e? Can you really split 19 up into 6.9897 parts, or even into $6\frac{1}{2}$ parts for that matter?

The sensible way to interpret the problem is that we do need a whole number of numbers so the best way forward seems to be to use 6 lots of e and the remaining number will then be $19 - 6\mathrm{e} \approx 2.69031$. The product of these seven numbers is then $\mathrm{e}^6 \times (19 - 6\mathrm{e}) \approx 1085.348$ but this is less than $\left(\frac{19}{7}\right)^7 \approx 1085.398$.

So does seven lots of $\frac{19}{7}$ generate the maximum product?